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Note on the representation of SU(n) and SU(n, 1) connected with the harmonic oscillator

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Abstract. One of the well known relations between SU(n) and the n dimensional isotropic harmonic oscillator is examined and extended to show exactly which representations of SU(n) can occur on the degeneracy spaces. The method is further extended to show which representations of SU(n, 1) can be generated by the oscillator problem.

1. Introduction

The relations between SU(n) and the n dimensional oscillator have been studied by many authors, such as Baker (1956), Demkov (1959), Jauch and Hill (1940), Dulock and MacIntosh (1965). Some present one or other of these relations as a proof that SU(n) is the symmetry group of the harmonic oscillator. But this leads to difficulties when considering the degeneracies of the anisotropic oscillator with rationally related frequencies and so the purpose of this paper is to examine one of these proofs in more detail than usual.

2. Representations generated by the harmonic oscillator

The argument in question depends on a connection between the creation and annihilation operators and the Lie algebra of SU(n). The hamiltonian H of the n dimensional isotropic harmonic oscillator is written as

$$H = \sum_{j=1}^{n} a_j^* a_j \tag{1}$$

where

$$[a_j, a_k^*] = \delta_{j,k}. \tag{2}$$

Then it is noted that there are certain combinations of the a_j and a_k^* which commute with the hamiltonian and which have the same commutation relations as the infinitesimal generators of SU(n). However, the result is more general than this, for consider all the operators of the form $a_j^* a_k$ where $j, k = 1, \ldots, n$.

Let $E_{j,k}$ be the $n \times n$ matrix with 1 in the (j, k) position and zero everywhere else, then, using (2) it is easy to show that each $a_j^* a_k$ commutes with the hamiltonian H defined in (1) and that the $a_j^* a_k$ satisfy the same commutation relations as the $E_{j,k}$. As the $E_{j,k}$ are a basis of the Lie algebra gl(n) derived from the group GL(n, C), this shows that each degeneracy space of H carries a representation of gl(n). It can also be shown that the $a_j^* a_k$ permute the eigenvectors of H, and so the representations of gl(n) are irreducible. Further, these representations can be exponentiated to give finite, analytic, irreducible representations of GL(n, C) on each degeneracy space of H. However, such representations are not necessarily unitary.

In fact, as the infinitesimal generators $a_j^* a_k$ of these representations are not anti-symmetric, the representations cannot be unitary.

As U(n) is a subgroup of GL(n, C), each degeneracy space of H carries a representation of U(n). The infinitesimal generators of these representations are represented by anti-symmetric combinations of the $a_j^* a_k$, so that the representations of U(n) must be unitary. A further restriction cannot affect the unitarity and these representations of GL(n, C) remain irreducible when restricted to SU(n). Thus it can be concluded that each degeneracy space of H carries an irreducible unitary representation of SU(n).

3. Representation of SU(n)

Although not essential, the final restriction to SU(n) is useful because the theory of Casimir operators can now be used to calculate which irreducible representations of SU(n) are realized on the degeneracy spaces by the method just outlined. The appropriate Casimir operators are the $C(n, d)$.

$$C(n, d) = \sum_{j,k,\ldots,m=1}^{n} X_k^j X^k \ldots X_m X_j^m, \tag{3}$$

where each summand is the product of d operators X_k^j and the sum is taken over all possible $j, k, \ldots, m$.

For the representations of SU(n) under consideration, X_k^j is represented by $a_j^* a_k$. Thus, in this realization, each summand in (3) is represented by $a_j^*(a_k a_k^* \ldots a_m a_m^*) . a_j$ and so

$$C(n, d) = \sum_j a_j^* \left(\sum_m a_m a_m^* \right)^{d-1} a_j.$$

Using (1) and (2) it can be shown that

$$\sum_m a_m a_m^* = H + n$$

and that

$$(H+n)a_j = a_j(H+n-1).$$

Hence, in this realization,

$$C(n, d) = \sum_{j=1}^{n} a_j^* a_j (H+n-1)^{d-1} = H(H+n-1)^{d-1}, \tag{4}$$

which means that $C(n, d)$ takes a fixed value on each degeneracy space of H. Indeed, if the value of H on one of its degeneracy spaces is E, then $C(n, d)$ takes the value $E(E+n-1)^{d-1}$ on this space.

But the values of $C(n, d)$ on each irreducible unitary representation of SU(n) have been calculated by Perelomov and Popov (1968). Let $\boldsymbol{m} = (m_1, \ldots, m_n)$, $m_{j-1} \geqslant m_j$, be the highest weight specifying an irreducible representation of SU(n), and $C(n, d)(\boldsymbol{m})$ the value of $C(n, d)$ on that representation, then they show that

$$\sum_{d=0}^{\infty} C(n, d)(\boldsymbol{m})z^d = z^{-1}\left\{1 - \prod_{j=1}^{n}\left(\frac{1-(\lambda_j+1)z}{1-\lambda_j z}\right)\right\} \tag{5}$$

where $\lambda_j = m_j + n - j$.

Suppose $\boldsymbol{m}$ characterizes a representation realized on the degeneracy space of H, on which H takes the value E, then (4) can be used to evaluate (5). In fact, remembering that $C(n, 0)(\boldsymbol{m}) = n$, and substituting the other values from (4) the left-hand side can be summed to give $n + Ez\{1 - z(E+n-1)\}^{-1}$ and (5) can be simplified to give the expression

$$\left(\prod_{j=1}^{n}(1-\lambda_j z)\right)\{1-(E+n)z\}\{1-(n-1)z\} = \{1-(E+n-1)z\}\left(\prod_{j=1}^{n}\{1-z(\lambda_j+1)\}\right). \tag{6}$$

Comparing the coefficients of z^{n+2} on either side of (6) and remembering that $E > 0$, $n \geqslant 2$, it is clear that for some j, $\lambda_j = 0$. Now the left-hand side of (6) is zero when $z = (E+n)^{-1}$, $(n-1)^{-1}$ or $\lambda_j^{-1}(\lambda_j \neq 0)$, whereas the right-hand side is zero when $z = (E+n-1)^{-1}$ or $(\lambda_j+1)^{-1}$, provided $(\lambda_j+1) \neq 0$. As these two sets of zeros must be identical, there must be a value of k such that $\lambda_k = 1$ for every value of j such that $\lambda_j = 0$. Proceeding in this way it can be shown that $\lambda_1, \ldots, \lambda_n$ is some permutation of $0, 1, \ldots, n-2, (E+n-1)$. Using the relations between λ_j, m_j and m_{j-1}, this implies that $\boldsymbol{m} = (E, 0, 0, \ldots, 0)$ which characterizes a representation belonging to the set of completely symmetric irreducible representations of SU(n).

Thus it has been shown that each degeneracy space of the hamiltonian of the isotropic harmonic oscillator carries a representation of GL(n, C) generated by the commutation relations. When restricted to SU(n) this irreducible representation is unitary and is the completely symmetric representation with highest weight $\boldsymbol{m} = (E, 0, 0, \ldots, 0)$.

4. Representation of SU(n, 1)

There is also an interesting connection between SU(n, 1) and the n dimensional harmonic oscillator, which can be demonstrated using the same sort of reasoning.

Take H, a_j and a_k^* as defined in (1) and (2), and consider the $(n+1)^2$ operators, $a_j^* a_k, \mathrm{i}\sqrt{H}a_k, \mathrm{i}a_j^*\sqrt{H}, -H$ where j and k take the values 1 to n. As H commutes with each $a_j^* a_k$, so does $\sqrt{H}$ and hence the commutation relations between these operators can be calculated. It turns out that these are the same as the commutation relations between the matrices $E_{j,k}$; $E_{n+1,k}$; $E_{j,n+1}$; $E_{n+1,n+1}$, and so the operators define a representation of the Lie algebra gl($n+1$). However, this is not a faithful representation of gl($n+1$) and must be restricted to sl($n+1$) if there is to be a 1–1 correspondence. If it is further restricted to the subalgebra su(n, 1) of sl($n+1$) then all the operators are skew-symmetric. Thus this restricted representation can be exponentiated to give a faithful unitary representation of SU(n, 1) realized on the same space as the oscillator problem.

From the previous discussion it is clear that when this representation is restricted to SU(n) it will decompose into the totally symmetric representations of SU(n) and will leave the degeneracy spaces of H invariant.

Again the Casimir operators, $C(n+1, d)$ can be used to show which representations of SU(n, 1) can be realized like this. In this case the $C(n+1, d)$ can be calculated by splitting them into several parts, depending on how many terms like X_j^{n+1} occur at either end of the summand. Let

$$A = \sum_{j,k,m} X_k^j X^k \dots X_j^m \qquad j \neq n+1$$

$$B_{r,s} = \sum_{j,m} \overbrace{X_{n+1}^{n+1} X_j^{n+1}}^{r \text{ terms}} \dots \overbrace{X_{n+1}^m X_{n+1}^{n+1}}^{1+s \text{ terms}} \qquad j, m \neq n+1.$$

Now $X_{n+1}^{n+1} = -H$ and both H and $\sqrt{H}$ commute with all the other terms which can occur so that if $l = r+s < d-2$, then $B_{r,s} = (-H)^l B_l$ where

$$B_l = \sum_{j,k=1}^{n} a_k \, . \, \overbrace{[X^k \dots X_j]}^{d-(l+1) \text{ terms}} \, . \, a_j^*.$$

For each l there are l possible pairs (r, s) and so

$$C(n+1, d) = A + \sum_{l=1}^{d-2} l(-H)^l B_l + (d-1)(H+n)(-H)^{d-1} + (-H)^d. \qquad (7)$$

Both A and B_l can be calculated by observing what happens when combinations of X_{n+1}^{n+1}, X_j^{n+1} and X_{n+1}^m occur in a summand and then seeing how often this can happen. Adding together all the summands which have exactly p pairs of indices equal to $n+1$, A can be split up into terms like $C(n, d-p)(1-H)^p$ multiplied by $\binom{d-1}{p}$. Using (4) these terms can be summed to show that

$$A = Hn^{d-1}.$$

Similarly it can be shown that

$$B_l = (n+H)^2 n^{d-l-2}.$$

Putting the values of A and B_l in the right-hand side of (7) and adding all the terms together, it turns out that $C(n+1, d)$ is identically equal to zero for all integer values of d in the range 1 to $n+1$.

Thus the n dimensional isotropic oscillator generates a faithful unitary representation of SU(n, 1) which is realized on the same space as the hamiltonian and whose Casimir operators are all zero. When this representation is restricted to SU(n) it decomposes into the completely symmetric representations, which leave the degeneracy spaces of the hamiltonian invariant.

5. Conclusions

It has been shown that the representations of SU(n) connected with the isotropic harmonic oscillator are the restrictions of representations of GL(n, C) also connected with the oscillator. The restriction to SU(n) means that the representations involved can be uniquely specified by evaluating the Casimir operators. Furthermore, there is a faithful representation of SU(n, 1) connected with the isotropic oscillator and for this all the Casimir operators take the value zero.

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